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# $P^*$-REDUCIBLE FINSLER SPACES AND APPLICATIONS

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## ABSTRACT

In this paper, we have considered special form of v (hv) -torsion tensor $P_{ijk} = \lambda c_{ijk} + \mu(h_{ij}c_k + h_{jk}c_i + h_{ki}c_j)$, where $\lambda$ & $\mu$ are scalar functions, positively homogeneous of degree one I $y^i$and call such a Finsler space is $P^*$-Reducible Finsler space. Also, we have worked out the role of $P^*$-Reducible Finsler spaces, in other special Finsler spaces.

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**KEYWORDS:** Finsler Space, C-Reducible, Semi C-Reducible, $S_3$ −Like $S_4$ −Like, Two &Three Dimensional Finsler Spaces

## 1. INTRODUCTION

In the present paper, we study $F^n$ be n-dimensional Finsler space, equipped with metric function $L(x, y)$. in the geometry of Finsler spaces, based on Cartan's connection; we have three kinds of covariant derivative of a tensor field. We shall denote $|_j$ as the h-covariant differentiation $|_j$, as the v-coariant differentiation and $\dot{\partial}_j$ as the $\delta$-differentaition, i.e. the partial differentiation, with respect to the element of support.

Nabil L. Youssef, S. H. Abed and A. Soleiman investigated intrinsically conformal changes, in Finsler geometry. Also, they studied the conformal change of Barthel connection and its curvature tensor, the conformal changes of Cartan and Berwald connections, as well as their curvature tensors. Shun-ichi Hojo, M. Matsumoto and K. Okubo discussed the theory of conformal Berwald Finsler spaces and its applications to $(\alpha, \beta) -$ metrics.

The notion of Douglas space has been introduced by M. Matsumoto and S. Bacso, as a generalization of Berwald space, from the view point of geodesic equations. It is remarkable that, a Finsler space is a Douglas space or is of Douglas type, if and only if the Douglas tensor vanishes identically.

M. Matsumoto studied on Finsler spaces, with $(\alpha, \beta) -$ metric of Douglas type. Hong-Suh Park and Eun-Seo Choi explained Finsler spaces, with an approximate Matsumoto metric of Douglas type. The authors S. Bacso and I. papp studied on a generalized Douglas space. Also, the team of authors Benling Li, Yibing Shen and Zhongmin Shen studied in a class of Douglas metrics.

## 2. PRELIMINARIES

Let $F^n$ be n-dimensional Finsler space, equipped with metric function $L(x, y)$. in the geometry of Finsler spaces, based on Cartan's connection, we have three kinds of covariant derivative of a tensor field. We shall denote $|_j$ as the h-covariant differentiation $|_j$, as the v-coariant differentiation and $\dot{\partial}_j$ as the $\delta$-differentaition, i.e. the partial differentiation with respect to the element of support. There are three curvature tensors and three torsion tensors of Cartan's connection $c\Gamma$. These are

1. $R_{hijk}$ h-curvature tensor,
2. $P_{hijk}$ hv-curvature tensor,
3. $S_{hijk}$ v-curvature tensor,
4. $R_{ijk} = Y^h R_{hijk}$ (v)h-torsion tensor,
5. $P_{ijk} = Y^h P_{hijk}$ (v)hv-torsion tensor, and
6. $C_{ijk} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k L^2$ (h)hv-torsion tensor

Various interesting forms of these curvature tensors and torsion tensors, have been studied by Matsumoto and others [8], [13], [14]. Two of them are C-reducible Finsler space and P-reducible Finsler space, in which the (h)hv-torsion tensor and (v)hv-torsion tensor are of the forms

$C_{ijk} = \frac{1}{n+1}(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j)$

$P_{ijk} = \frac{1}{n+1}(h_{ij}P_k + h_{jk}P_i + h_{ki}P_j)$ Respectively,

Here, $h_{ij}$ is the angular metric tensor, $C_i = C_{ijk}g^{jk}$ and $P_i = P_{ijk}g^{jk}$. It is to be noted that, fundamental function of any C-reducible Finsler space is of the Randers type or the Kropina type [12]. Every C-reducible Finsler space is P-reducible and converse is not necessarily true. In 1976 H. Izumi [1], [2] introduced a $P^*$-Finsler space, in which $P_{ijk}$ is of the form $P_{ijk} = \lambda C_{ijk}$ , where $\lambda$ is a scalar function homogeneous of degree zero in $y^i$. It should also be noted that, in a c-concircularly flat Finsler space, the $P_{ijk}$ may be written in this form.

The Purpose of the Present Paper is to Consider a Special Form of $\boldsymbol{P_{ijk}}$, Given By

$$P_{ijk} = \lambda c_{ijk} + \mu\left(h_{ij}c_k + h_{jk}c_i + h_{ki}c_j\right) \qquad \textbf{(2.1)}$$

where$\lambda$ and $\mu$are scalar functions, positively homogeneous of degree one in $y^i$. We shall call such a Finsler space as $P^*$-reducible Finsler space. We will obtain the role of $P^*$-reducible Finsler spaces in different special Finsler spaces.

A Finsler space with $P_{ijk}$ of the form (2.1), reduces to a $P^*$-Finsler space, when $\mu$ vanishes, whereas it reduces to a P-reducible Finsler space, when $\lambda$ vanishes. Thus a Finsler space satisfying (2.1) is a generalized form of $P^*$ and P-reducible Finsler space.

That is why, we say a Finsler space satisfying (2.1) as $P^*$ -reducible Finsler space.

A Finsler space with $P_{ijk} = 0$ is called a Landsberg space [14]. If $C_{ijk|h} = 0$, then $F^n$ is called Berwald's space [10], [16], whereas $F^n$ is called h-isotropic Finsler space [5], if its h-curvature tensor $R_{hijk}$ is written in the form

$R_{hijk} = R(g_{hj}g_{ik} - g_{hk}g_{ij})$. Where R is non-zero scalar.

We quote the following results, which will be used in this paper.

**Lemma 2.1**

[8] If the hv-curvature tensor $P_{hijk}$ of a c-reducible Finsler space vanishes then it is a Berwald's space.

**Lemma 2.2**

[7]A Finsler space $F^n$ is locally Minkowskian iff h-curvature tensor $R_{hijk} = 0$ and $C_{ijk|h} = 0$.

**Lemma 2.3**

[15] In an h-isotropic Finsler space $P_{hijk} = R_{hikj}$and $S_{hijk} = 0$.

## 3. n-DIMENSIONAL $P^*$-REDUCIBLE FINSLER SPACE

If$F^n$ is a Landsberg space then$P_{ijk} = 0$. Therefore from (2.1), we have

$c_{ijk} = -\mu\left(h_{ij}c_k + h_{jk}c_i + h_{ki}c_j\right)$, and $(n+1)\mu = \lambda$ , hence we have the following

**Theorem 3.1**

If$P^*$-reducible Finsler space is a Landsberg space, and then it is a c-reducible Finsler space.

Since, in a Landsberg space$P_{hijk} = 0$, we have from lemma 2.1

**Theorem 3.2**

If$P^*$-reducible Finsler space is a Landsberg space, then it is a Berwald space.

In virtue of Lemma 2.2, we have the following-

**Theorem 3.3**

If $P^*$-reducible Finsler space is a Landsberg space with vanishing h-curvature tensor, then it is a locally Minkowskian space.

To find the hv-curvature tensor, we use

$$P_{hijk} = \psi_{(hi)}\{P_{ijk|h} + P_{ikr}c^r_{jk}\} \quad \textbf{(3.1)}$$

$$S_{hijk} = \psi_{(hi)}\{c_{hkr}c^r_{ij}\}$$

And $h_{ij|h} = -L^{-1}(h_{ih}l_j + h_{jh}l_i)$

Where $\psi_{(hi)}\{\dots\}$ denotes the interchange of indices h,i and subtraction. Thus, for a $P^*$-Reducible Finsler space, we have

$$P_{hijk} = \psi_{(hi)}\{a_h c_{ijk} + \mu_h c_i h_{jk} + E_{jh}h_{ik} + F_{kh}h_{ij}\} - \lambda s_{hijk} \quad \textbf{(3.2)}$$

where

$a_h = \lambda_h - \mu c_h$ ,

$E_{jh} = c_j\mu_h + \mu\dot{\partial}_h c_j + \mu L^{-1}(l_h c_j + l_j c_h)$ ,

$F_{kh} = \mu_h c_k + \mu c_{k|h} + \mu L^{-1}(l_k c_h + l_h c_k)$

$\lambda_h = \dot{\partial}_h \lambda$ and $\mu_h = \dot{\partial}_h \mu$.

A $P_2$-Like Finsler space has been introduced by Matsumoto [6], in which $P_{hijk}$ of the form

$$P_{hijk} = \psi_{(hi)}\{a_h c_{ijk}\} \tag{3.3}$$

On the other hand, he [6] also studied a Finsler space, in which $P_{hijk}$ is of the form

$$P_{hijk} = G_{ih} h_{jk} + \psi_{(hi)}\{E_{jh} h_{ik} + F_{kh} h_{ij}\} \tag{3.4}$$

where$a_i, G_{ih}, E_{jh}$and$F_{kh}$ are Finsler tensor fields. The form (3.1) of $P_{hijk}$ shows that it is more general than the form (3.2) and (3.3) of $P_{hijk}$.

Since the curvature tensors $P_{hijk}$and$S_{hijk}$ have the identities,

$$-S_{hijk|o} = P_{hijk} - P_{hikj}, \tag{3.5}$$

$$S_{hijk} = -S_{ihjk} = -S_{hikj} = P_{jkhi} \tag{3.6}$$

and$S_{hijk} + S_{hjki} + S_{hkij} = 0$ **(3.7)**

Therefore from (3.2), we get

**Theorem 3.4**

In $P^*$-reducible Finsler space,

i. $S_{hijk|o} = \psi_{(hi)}\{\mu c_r c^r_{kh} h_{ij} - \mu c_r c^r_{jh} h_{ik} + 2\lambda s_{hijk}\}$ **(3.8)**

ii. $P_{hijk} + P_{ikjh} + P_{khji} = 0$ **(3.9)**

**Remark**

It should be noted that, the second identity in theorem (3.5) also holds in a P-reducible Finsler space.

Next, we consider a P-symmetric Finsler space, in which $P_{hijk} = P_{hikj}$. In view of (3.5) and (2.7), we get

$$2\lambda S_{hijk} = \mu c_r \psi_{(hi)}\{c^r_{jh} h_{ik} - c^r_{kh} h_{ij}\} \tag{3.10}$$

In an $S_4$-like Finsler space thev-curvature tensor $S_{hijk}$ is of the form

$$L^2 S_{hijk} = h_{hj} M_{ik} + h_{ik} M_{hj} - h_{hk} M_{ij} - h_{ij} M_{hk} \tag{3.11}$$

Where $M_{ij}$ is a symmetric indicatory tensor.

From (3.10) and (3.11), we have the following,

**Theorem 3.5**

A $P^*$-Reducible P-symmetric Finsler space is an $S_4$-Like Finsler space.

Now let us suppose that $F^n$ is h-isotropic, then in view of lemma 3, it is P-symmetric and $S_{hijk} = 0$. Hence from (3.10), we get $\mu = 0$, as $\psi_{(hi)}\{c^r_{jh} h_{ik} - c^r_{kh} h_{ij}\} \neq 0$.

Thus

We have the following.

**Theorem 3.6**

If a $P^*$-reducible P-symmetric Finsler space is h-isotropic, then it is a $P^*$-Finsler space.

Now let us suppose that $F^n$ admits a concurrent vector field $x^i$, then $x^i_{|j} = -\delta^i_j$and $x^i_{|j} = 0$, which leads to

$x^h P_{ijk} + c_{ijk} = 0.$

Transvecting this equation with $y^i$ and $x^i$ respectively, we get

$x^h P_{hik} = 0, c_{hij} x^h = 0.$

In view of (2.1) these equations lead to

$\mu\{h_{ij} x^i c_k + h_{ki} x^i c_j\} = 0$

Since $\{h_{ij} x^i c_k + h_{ki} x^i c_j\} \neq 0$ for a concurrent vector field $x^i$, therefore $\mu = 0$.

**Theorem 3.7**

If a $P^*$-reducible Finsler space admits a concurrent vector field, then it is a $P^*$-Finsler space.

There are two expressions for the hv-curvature tensor $P_{hijk}$ , one of them is given by (3.1), whereas the other is

$$P_{hijk} = \psi_{(hi)}\{c_{ijk/h} + c^r_{hj} P_{rik}\} \quad \textbf{(3.12)}$$

Substituting (2.1) into the two relations (3.1) and (3.11), we have the following theorem

**Theorem 3.8**

A $P^*$-reducible Finsler space satisfies

$$c_{hkl/i} - c_{ikl/h} = \psi_{(hi)}\{\lambda_i c_{hkl} + \mu_i(h_{hk} c_l + h_{kl} c_h + h_{lh} c_k) + \mu(h_{hk} c_l|_i + h_{lh} c_k|_i + 2c_i c_{hkl}) + \mu L^{-1}\big(h_{hk}(l_i c_l + l_l c_i) + h_{lh}(l_k c_i + l_i c_k) + \mu c^r(h_{ik} c_{rhi} + h_{li} c_{rhk})\big)\}, \quad \textbf{(3.13)}$$

$$c_{h|i} - c_{i|h} = \psi_{(hi)}\{\lambda_i c_h + (n+1)\mu_i c_k\}. \quad \textbf{(3.14)}$$

Since the scalars $\lambda$ and $\mu$ are homogeneous function of degree one in $y^i$, we can easily show

**Theorem 3.9**

The condition that, a $P^*$-reducible Finsler space be a Landsberg space is that,

$\lambda_i c_h + \lambda_h c_i = 0 \; and \; \mu_i c_h - \mu_h c_i = 0.$

## 4. THREE DIMENSIONAL $P^*$-REDUCIBLE FINSLER SPACE

First of all, we shall discuss the two dimensional Finsler space $F^2$. With reference to Berwald's frame $(l_i, m_i)$, the angular metric tensor, (h) hv-torsion tensor and (v) hv-torsion tensors are given by

$h_{ij} = m_i m_j, L c_{ijk} = I m_i m_j m_k$ and

$P_{ijk} = I_{|o} m_i m_j$ **(4.1)**

Where I am the main scalar, from above equations it follows that, $P_{ijk}$ may be written as the from $LI_{|o} = I(\lambda + 3\mu)$. Thus we have the following-

**Theorem 4.1**

Every three dimensional Finsler space is $P^*$-Reducible Finsler space, where the main scalar $I$ satisfies

$LI_{|o} = I(\lambda + 3\mu)$.

Next we deal with bi-two dimensional Finsler space. With respect to orthonormal frame $e^i_{(\alpha)}, \alpha = 0,1,2,3,$ the (h) hv-torsion tensor has written as

$Lc_{ijkl} = c_{\alpha\beta\gamma\eta} e_{(\alpha)i} e_{(\beta)j} e_{(\gamma)k} e_{(\eta)l}$ **(4.2)**

Where the scalar component $e_{\alpha\beta\gamma\delta}$ are such that $c_{o\beta\gamma\delta} = 0$.

$c_{1111} = H, c_{1222} = I, c_{2222} = -c_{1112} = J, c_{1122} = K$

The scalars $H, I, J$ and $K$ are called main scalars and satisfies the equation $H + I + J = Lc$.

If $c_{\alpha\beta\gamma\eta,\delta}$ are h-scalar derivative of $c_{\alpha\beta\gamma\eta}$ , then we have,

$c_{o\beta\gamma\eta,\delta} = 0, c_{1111,\delta} = H_{,\delta} + 3Jh_\delta,$

$c_{1222} = -J_{,\delta} + (H - 2I)h_{\delta,}$

$c_{1112,\delta} = I_{,\delta} - 3Jh_\delta$and

$c_{2222,\delta} = J_\delta + 3Ih_{\delta,}$

Where$h_\delta$ are adapted components of h-connection vector$h_j$.

From (4.2), it follows that

$P_{ijkl} = c_{\alpha\beta\gamma\eta,0} e_{(\alpha)i} e_{(\beta)j} e_{(\gamma)k} e_{(\eta)l}$

Since $(\delta_{\alpha\beta\gamma} - \delta_{0\alpha\beta}\delta_{0\beta\gamma})$ are scalar components of$h_{ij}$and $c_i = ce_{(1)i}$, therefore from (2.1), (4.2), (4.3) and (4.4), we get

$H_{,0} + 3Jh_0 = \lambda H + 3\mu c$ ,

$J_{,0} + 3Ih_0 = \lambda J$ ,

$-J_{,0}Hh_0 - 2Ih_0 = -\lambda J$ ,

$I_{,0} - 3Jh_0 = \lambda I + \mu c$.

Solving these equations, we get

$h_0 = 0, \lambda = \frac{J_{,0}}{J}$ *and* $\mu = \frac{I_{,0}}{c} - \frac{J_{,0}I}{cJ}$

Hence we have the following-

## CONCLUSIONS

The special Finsler spaces can be applied in various branches of theoretical & computational Physics, theory of anisotropic media, Lagrangian mechanics, to solve optimization problems, theory of Ecology, theory of evolution of Biological Systems, in describing the internal symmetry of Hedrons, theory of Space Time & Gravitation, deformation of crystalline media, Seismic Phenomena, interfaces in thermodynamics system etc.

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